

# Borel stability for congruence subgroups

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## Abstract

We prove homological stability theorems for congruence subgroups of general linear groups and symplectic groups. From these theorems, we deduce a generalization of a theorem of Borel showing that certain homology groups of a congruence subgroup do not depend on the level of the congruence subgroup.

## 1 Introduction

A theorem of Borel [2, 7.5] states that for a semisimple real Lie group  $G$ , the rational group homology  $H_m(\Gamma; \mathbb{Q})$  is the same for any lattice  $\Gamma \subset G$ , at least when  $m$  is smaller than some constant depending on  $G$ . Consider the case where  $G$  is the special linear group  $SL_n$ . Given an integer  $p \geq 2$ , define the level  $p$  congruence subgroup  $SL_n(\mathbb{Z}, p\mathbb{Z})$  to be the kernel of the natural map  $SL_n(\mathbb{Z}) \rightarrow SL_n(\mathbb{Z}/p\mathbb{Z})$ . Then for each  $m$ , Borel [2, §9] showed that there exists an integer  $N_m$  such that  $H_m(SL_n(\mathbb{Z}, p\mathbb{Z}); \mathbb{Q}) \cong H_m(SL_n(\mathbb{Z}); \mathbb{Q})$  for all  $m > N_m$ . We wish to generalize this result to congruence subgroups defined over rings other than  $\mathbb{Z}$ . Please note that throughout the rest of the paper, we only consider two-sided ideals.

**Definition.** For an arbitrary ring  $R$  and an ideal  $\mathfrak{a} \subset R$ , the congruence subgroup  $GL_n(R, \mathfrak{a})$  is  $\ker(GL_n(R) \rightarrow GL_n(R/\mathfrak{a}))$ .

We prove (Theorem 6.1) the following natural generalization of Borel's theorem.

**Theorem** (Borel stability for congruence subgroups). *Suppose that  $R$  is a commutative Noetherian ring of Krull dimension  $k - 1$  and  $\mathfrak{a}$  is an ideal of  $R$  with  $R/\mathfrak{a}$  finite. Then for  $m \geq 1$  and  $n \geq 2m + 2k + 4$ , the map  $H_m(GL_n(R, \mathfrak{a}); \mathbb{Q}) \rightarrow H_m(GL_n(R); \mathbb{Q})$  is an isomorphism.*

We also prove a version of this theorem for symplectic groups (Theorem 6.2).

**Definition.** Suppose  $R$  is a commutative ring and  $\mathfrak{a}$  an ideal of  $R$ . The symplectic congruence subgroup  $\mathrm{Sp}_{2n}(R, \mathfrak{a})$  is  $\ker(\mathrm{Sp}_{2n}(R) \rightarrow \mathrm{Sp}_{2n}(R/\mathfrak{a}))$ .

**Theorem** (Borel stability for congruence subgroups of symplectic groups). *Suppose that  $R$  is a commutative Noetherian ring of Krull dimension  $k - 2$  and  $\mathfrak{a}$  is an ideal of  $R$  with  $R/\mathfrak{a}$  finite. Then for  $m \geq 1$  and  $n \geq 2m + 2k + 4$ , the map  $H_m(\mathrm{Sp}_{2n}(R, \mathfrak{a}); \mathbb{Q}) \rightarrow H_m(\mathrm{Sp}_{2n}(R); \mathbb{Q})$  is an isomorphism.*

In proving these theorems, we employ the method outlined by Putman [11, 5.11]: we first prove a homological stability result for congruence subgroups and then use the stability trick introduced in [12, Lemma 5.1]. A sequence of groups  $G_1 \subset G_2 \subset G_3 \subset \dots$  is said to be homologically stable if the map  $H_m(G_{n-1}) \rightarrow H_m(G_n)$  induced by inclusion  $G_{n-1} \rightarrow G_n$  is an isomorphism for all  $n > N_m$ , where  $N_m$  is a constant depending on  $m$ . This property is known to hold for many natural sequences of groups including symmetric groups [10], general linear groups over rings satisfying stable range conditions [14], and mapping class groups [6]. However, it does not hold for congruence subgroups. Consider the case where  $G_n$  is the congruence subgroup  $\mathrm{SL}_n(\mathbb{Z}, p\mathbb{Z})$ . Then Lee and Szczarba [8] showed that  $H_1(G_n) \cong \mathfrak{sl}_n(\mathbb{Z}/p\mathbb{Z})$  for  $n \geq 3$ , so homological stability does not hold. However, if we take homology with rational rather than integral coefficients, then homological stability does hold, i.e., for all  $m$  there exists  $N_m$  such that  $H_m(G_{n-1}; \mathbb{Q}) \rightarrow H_m(G_n; \mathbb{Q})$  is an isomorphism for  $n > N_m$ , by the aforementioned work of Borel [2]. See Putman [13] for a description of how  $H_m(G_n; \mathbb{Z}/p\mathbb{Z})$  changes as  $n$  increases.

More generally, for  $\mathrm{GL}_n(R, \mathfrak{a})$ , Charney [4] proved that with appropriate coefficients, homological stability will hold for a wide class of rings  $R$ , for instance any commutative Noetherian ring of finite Krull dimension. Charney's proof, like most proofs of homological stability theorems, follows a technique introduced in the unpublished work of Quillen which involves examining the action of congruence subgroups on various simplicial complexes. In this paper, we simplify Charney's argument, then apply it to congruence subgroups of symplectic groups. In particular, we prove the following, as Theorems 4.3 and 5.3.

**Theorem** (Charney). *Suppose that  $R$  is a commutative Noetherian ring of Krull dimension  $k - 1$ , and  $\mathfrak{a}$  an ideal of  $R$ . Suppose further that the inclusion*

of groups  $\mathfrak{a} \subset R$  induces an isomorphism on homology (with coefficients in  $M$ ):  $H_*(\mathfrak{a}; M) \xrightarrow{\cong} H_*(R; M)$ . Then for  $n \geq 2m + k + 4$ , the map

$$H_m(\mathrm{GL}_{n-1}(R, \mathfrak{a}); M) \rightarrow H_m(\mathrm{GL}_n(R, \mathfrak{a}); M)$$

is a surjection, and for  $n \geq 2m + k + 5$ , this map is an isomorphism.

**Theorem.** Suppose that  $R$  is a commutative Noetherian ring of Krull dimension  $k - 2$ , and  $\mathfrak{a}$  an ideal of  $R$ . Suppose further that the inclusion of groups  $\mathfrak{a} \subset R$  induces an isomorphism on homology (with coefficients in  $M$ ):  $H_*(\mathfrak{a}; M) \xrightarrow{\cong} H_*(R; M)$ . Then for  $n \geq 2m + k + 4$ , the map

$$H_m(\mathrm{Sp}_{2(n-1)}(R, \mathfrak{a}); M) \rightarrow H_m(\mathrm{Sp}_{2n}(R, \mathfrak{a}); M)$$

is a surjection, and for  $n \geq 2m + k + 5$ , this map is an isomorphism.

From these theorems, and the above mentioned stability trick, we will derive our Borel stability theorems for congruence subgroups of linear and symplectic groups. It is worth noting that we use no analytic techniques, in contrast to Borel.

**Outline.** §2 reviews the basic equivariant homology needed to operate Quillen's machine for proving homological stability theorems. §3 proves a generic theorem which states that a sequence of groups enjoys homological stability if it acts on a sequence of complexes in an appropriate way. As usual, the proof proceeds by comparing certain spectral sequences which converge to the equivariant homology of these complexes. §4 more or less reproduces Charney's result by showing that our generic theorem applies to congruence subgroups of general linear groups over rings satisfying stable range conditions acting on complexes of split unimodular sequences. §5 shows that the generic theorem can also be applied to congruence subgroups of symplectic groups over rings satisfying symplectic stable range conditions acting on complexes of hyperbolic pairs. Finally §6 uses our results to obtain our Borel stability results.

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## 2 Equivariant Homology

In the next section, we will prove a theorem stating that whenever a sequence of groups acts on a sequence of (simplicial) complexes in a prescribed way, the groups enjoy homological stability. To do this, we must extensively exploit the theory of equivariant homology, so we review this theory now (a lucid reference is [3, §VII.7]).

**Definition.** Suppose a group  $G$  acts on the left on a simplicial complex  $X$ . Then we can form the equivariant homology of  $G$  acting on  $X$ , denoted  $H_*^G(X)$ , by taking the homology of the chain complex  $F_* \otimes_G C_*(X)$  where  $F_*$  is a free resolution of  $\mathbb{Z}$  over  $\mathbb{Z}[G]$  and  $C_*(X)$  is the simplicial chain complex of  $X$  (which carries a natural left  $G$  action).

Here, the operation  $\otimes_G$  is a slightly more subtle version of tensoring over  $\mathbb{Z}[G]$ : in particular, if  $A$  and  $B$  are both left  $G$ -modules, then  $A \otimes_{\mathbb{Z}} B$  can be given the diagonal left  $G$ -action  $g \cdot (x \otimes y) = gx \otimes gy$ , so we can define  $A \otimes_G B = (A \otimes B)/G$ . Equivalently, we define a right module structure on  $A$  via  $x \cdot g = g^{-1} \cdot x$ , and take  $A \otimes_{\mathbb{Z}[G]} B$ . Of course, we can also compute equivariant homology with coefficients in some abelian group  $M$ : that is, we define  $H_*^G(X; M)$  to be the homology of  $F_* \otimes_G C_*(X, M)$ . Similarly, we define the reduced equivariant homology  $\tilde{H}_*^G(X; M)$  to be the homology of  $F_* \otimes_G \tilde{C}_*(X, M)$ .

For our purposes, this construction derives its importance from the fact that it can be computed via a spectral sequence which encodes information about the homology of  $G$  and its subgroups. Thus, computing  $\tilde{H}_*^G(X; M)$  will yield crucial implications for group homology. The main tool used in this computation is the following lemma [3, VII.7.3].

**Lemma 2.1.** *Suppose  $\tilde{H}_q(X; M) = 0$  for  $q = 0, \dots, m$ . Then  $\tilde{H}_q^G(X; M) = 0$  for  $q = 0, \dots, m$ .*

**The spectral sequence.** There is a spectral sequence [3][VII.7.7] with

$$\mathcal{E}_{pq}^1 = \bigoplus_{\sigma^p \in \Sigma^p} H_q(\text{Stab}_G \sigma^p)$$

where  $\Sigma^p$  is a collection of representatives of  $G$ -orbits of  $p$ -cells of  $X$ , and by convention there is a single  $-1$ -cell (since  $\tilde{C}_{-1}(X; M) = M$ ). This spectral sequence converges to

$$\tilde{H}_*^G(X, M).$$

If  $G$  acts “without rotations”, meaning that the stabilizer of any cell of  $X$  fixes the cell pointwise, then the differential  $d^1 : \mathcal{E}_{pq}^1 \rightarrow \mathcal{E}_{(p-1)q}^1$  can be readily described [3][VII.8]. Suppose  $\hat{\tau}^{p-1}$  is a face of  $\sigma^p \in \Sigma^p$ , and  $\tau^{p-1} \in \Sigma^{p-1}$  is the representative of its  $G$ -orbit, so we can write  $\tau^{p-1} = g\hat{\tau}^{p-1}$  with  $g \in G$ . Then because  $G$  acts without rotations:

$$\text{Stab}_G \sigma^p \subset \text{Stab}_G \hat{\tau}^{p-1} = g^{-1}(\text{Stab}_G \tau^{p-1})g$$

We conclude that inclusion followed by conjugation induces a natural map on homology:

$$d_{\sigma\tau}^1 : H_*(\text{Stab}_G \sigma^p; M) \rightarrow H_*(\text{Stab}_G \tau^{p-1}; M)$$

The differential  $d^1 : \mathcal{E}_{pq}^1 \rightarrow \mathcal{E}_{(p-1)q}^1$  is then just given by summing all of the  $d_{\sigma\tau}^1$ .

### 3 Spectral sequences

We will now apply Quillen’s machine to prove a “generic” homological stability theorem. Fix an abelian group  $M$ . From now on, all homology will be taken with coefficients in  $M$  unless otherwise noted, i.e.,  $H_*(X)$  means  $H_*(X; M)$ . Now suppose we have a sequence of groups  $G_1 \subset G_2 \subset \dots$ , and for each  $i$  we have a subgroup  $\Gamma_i \triangleleft G_i$  such that  $\Gamma_1 \subset \Gamma_2 \subset \dots$ . Suppose further that there are simplicial complexes  $X_1, X_2, \dots$  such that  $G_i$  acts on  $X_i$ . Let  $d$  be a positive integer.

**Theorem 3.1.** *Assume:*

1.  $G_n$  acts transitively on the  $p$ -cells of  $X_n$  for  $n > p + d$ .
  2. For all  $p = 0, \dots, n$ , there is a standard  $p$ -cell  $\sigma_0^p$  in  $X_n$  such that for  $n < p + d$  we have  $\text{Stab}_{\Gamma_n} \sigma_0^p = \Gamma_{n-p-1}$  and  $\text{Stab}_{\Gamma_n} \sigma_0^p$  fixes the vertices of  $\sigma_0^p$ .
  3.  $G_n$  acts trivially on the image of the map  $H_p(\Gamma_{n-1}) \rightarrow H_p(\Gamma_n)$  induced by inclusion  $\Gamma_{n-1} \rightarrow \Gamma_n$  for  $n > d$  and  $p \geq 0$ .
  4.  $X_n$  and  $X_n/\Gamma_n$  are  $\frac{n-d}{2}$  acyclic.
- Then the map  $H_m(\Gamma_{n-1}) \rightarrow H_m(\Gamma_n)$  is a surjection for  $n \geq 2m + d + 1$  and an isomorphism for  $n \geq 2m + d + 2$ .

*Proof.* We will proceed by induction (the prototype for our argument is [7, §5].) When  $m = 0$ , we have that  $H_m(\Gamma_{n-1}) \rightarrow H_m(\Gamma_n)$  is just  $M \xrightarrow{id} M$  which is clearly an isomorphism. Now suppose that  $m \geq 1$  and for all  $q < m$ ,  $H_q(\Gamma_{n-1}) \rightarrow H_q(\Gamma_n)$  is a surjection for  $n \geq 2q + d + 1$  and an isomorphism

for  $n \geq 2q + d + 2$

Recall our spectral sequence converging to  $\tilde{H}_*^{\Gamma_n}(X_n)$ . We have

$$\mathcal{E}_{pq}^1 = \bigoplus_{\sigma^p \in \Sigma^p} H_q(\text{Stab}_{\Gamma_n} \sigma^p),$$

where  $\Sigma^p$  is a collection of representatives of  $\Gamma_n$ -orbits of  $p$ -cells of  $X_n$ , (again, by convention there is a single  $-1$ -cell.) Most of our assumptions allow us to understand this  $\mathcal{E}_{pq}^1$  for small  $p$ . For instance, observe that by assumptions 1 and 2, if  $p < n - d$ , then  $\mathcal{E}_{pq}^1 \cong C_p(X_n/\Gamma_n; H_q(\Gamma_{n-p-1}))$  (since for each  $\sigma^p$ , assumption 1 yields some  $g \in G_n$  such that  $g\sigma_0^p = \sigma^p$ , and thus  $\text{Stab}_{\Gamma_n} \sigma^p = g \text{Stab}_{\Gamma_n} \sigma_0^p g^{-1} \cong \Gamma_{n-p-1}$  by assumption 2.) Observe also that following [3, §7.8], we can use assumptions 2 and 3 to conclude that for  $p + 1 < n - d$  there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_{pq}^1 & \xleftarrow{d_1} & \mathcal{E}_{(p+1)q}^1 \\ \cong \downarrow & & \cong \downarrow \\ \tilde{C}_p(X_n/\Gamma_n; H_q(\Gamma_{n-p-1})) & \xleftarrow{\phi} & \tilde{C}_{p+1}(X_n/\Gamma_n; H_q(\Gamma_{n-p-2})) \end{array}$$

where we can describe  $\phi$  as the composition of the boundary map

$$\tilde{C}_{p+1}(X_n/\Gamma_n; H_q(\Gamma_{n-p-2})) \rightarrow \tilde{C}_p(X_n/\Gamma_n; H_q(\Gamma_{n-p-2}))$$

and the natural map

$$\tilde{C}_p(X_n/\Gamma_n; H_q(\Gamma_{n-p-2})) \rightarrow \tilde{C}_p(X_n/\Gamma_n; H_q(\Gamma_{n-p-1})).$$

Now suppose  $n \geq 2m + d + 1$ . We know that, given a complex  $Y$  and a map of abelian groups  $f : P \rightarrow Q$ , the composition

$$\tilde{C}^0(Y; P) \rightarrow \tilde{C}^{-1}(Y; P) = P \xrightarrow{f} Q$$

can be surjective only if  $f$  is surjective. Therefore, to show that  $H_m(\Gamma_{n-1}) \rightarrow H_m(\Gamma_n)$  is surjective, it suffices to show that  $\mathcal{E}_{0m}^1 \rightarrow \mathcal{E}_{-1m}^1$  is surjective, which will follow if we can show that  $\mathcal{E}_{-1m}^2 = 0$ . Now, by 2.1 we have  $\tilde{H}_p^{\Gamma_n}(X_n) = 0$  for  $q \leq m$  since  $X_n$  is at least  $m$ -acyclic by assumption 4. Therefore we have  $\mathcal{E}_{-1m}^\infty = 0$ . Now we will use our inductive assumption to show that  $\mathcal{E}_{(m-q)q}^2 = 0$  for  $q < m$ . This will establish that for  $r \geq 2$ , every

differential to or from  $\mathcal{E}_{-1m}^r$  will be zero, and thus  $\mathcal{E}_{-1m}^2 = \mathcal{E}_{-1m}^\infty = 0$  as desired.

Suppose first that  $q \leq m - 2$  (as we will see, the case where  $q = m - 1$  is slightly harder). Then since

$$n - (m - q) - 1 \geq 2m + d + 1 - m + q - 1 = m + d + q \geq (q + 2) + d + q = 2q + d + 2$$

we have that

$$H_q(\Gamma_{n-m-q-2}) \xrightarrow{\cong} H_q(\Gamma_{n-m-q-1}) \xrightarrow{\cong} H_q(\Gamma_{n-m-q}) \rightarrow \dots$$

Writing  $Y_n = X_n / \Gamma_n$  and  $\Gamma = \Gamma_{n-m-q-2}$ , we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{E}_{-1q}^1 & \xleftarrow{d_1} & \dots & \xleftarrow{d_1} & \mathcal{E}_{(m-q+1)q}^1 \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \tilde{C}_{-1}(Y_n; H_q \Gamma) & \xleftarrow{\partial} & \dots & \xleftarrow{\partial} & \tilde{C}_{(m-q+1)}(Y_n; H_q \Gamma) \end{array}$$

By acyclicity of  $Y_n$ , the bottom row is exact, and thus  $\mathcal{E}_{(m-q)q}^2 = 0$  as desired. When  $q = m - 1$ , we have the same diagram, but we must take  $\Gamma = \Gamma_{n-m-q-1}$ , and the last downward arrow is merely a surjection. By the same logic we still have  $\mathcal{E}_{(m-q)q}^2 = 0$ , as desired.

Now we consider injectivity. Suppose  $n \geq 2m + d + 2$ . In order to show that  $H_m(\Gamma_{n-1}) \rightarrow H_m(\Gamma_n)$  is injective, we will first show that  $\mathcal{E}_{0m}^2 = 0$ , and then show that this implies that  $d_1 : \mathcal{E}_{0m}^1 \rightarrow \mathcal{E}_{-1m}^1$  is injective. Showing that  $\mathcal{E}_{0m}^2 = 0$  is exactly as before: we show that  $\mathcal{E}_{0m}^\infty = 0$ , and then that  $\mathcal{E}_{(m+1-q)q}^2 = 0$  for  $q < n$ , which implies that  $\mathcal{E}_{0m}^2 = \mathcal{E}_{0m}^\infty$ . Since  $n \geq 2m + d + 2$ , we know by assumption 4 that  $X_n$  is  $m + 1$ -acyclic, so  $E_{pq}^2 = 0$  for  $q \leq m + 1$ , therefore  $\tilde{H}_{m+1}^{\Gamma_n}(X_n) = 0$ , and thus  $\mathcal{E}_{0m}^\infty = 0$ . Now suppose  $q < m$ , we have

$$\begin{aligned} n - (m + 1 - q) - 1 &\geq 2m + d + 2 - m - 1 + q - 1 = m + d + q \\ &\geq q + 1 + d + q = 2q + d + 1 \end{aligned}$$

and thus  $H_q(\Gamma_{n-(m+1-q)-2}) \rightarrow H_q(\Gamma_{n-(m+1-q)-1})$  is a surjection and

$$H_q(\Gamma_{n-(m+1-q)-1}) \xrightarrow{\cong} H_q(\Gamma_{n-(m+1-q)}) \xrightarrow{\cong} H_q(\Gamma_{n-(m+1-q)+1}) \rightarrow \dots$$

So, writing  $\Gamma = \Gamma_{n-(m+1-q)-1}$ , and  $Y_n = X_n/\Gamma_n$ , we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{E}_{-1q}^1 & \xleftarrow{d_1} & \dots & \xleftarrow{d_1} & \mathcal{E}_{(m+1-q+1)q}^1 \\
\cong \downarrow & & \cong \downarrow & & \downarrow \\
\tilde{C}_{-1}(Y_n; H_q \Gamma) & \xleftarrow{\partial} & \dots & \xleftarrow{\partial} & \tilde{C}_{(m-q+1)}(Y_n; H_q \Gamma)
\end{array}$$

where the last downward arrow is surjective. By acyclicity of  $Y_n$ , the bottom sequence is exact, and it follows that  $\mathcal{E}_{(m+1-q)q}^2 = 0$  as promised, and we have succeeded in showing that  $\mathcal{E}_{0m}^2 = 0$ . On the other hand, the image of  $d_1 : \mathcal{E}_{1m}^1 \rightarrow \mathcal{E}_{0m}^1$  lies inside the kernel of the augmentation map  $\mathcal{E}_{0m}^1 = \tilde{C}_0(X_n/\Gamma_n; H_m(\Gamma_{n-1})) \xrightarrow{\text{aug}} H_m(\Gamma_{n-1})$ . In particular, it has zero intersection with the  $H_m(\Gamma_{n-1})$  summand corresponding to (the orbit of) our canonical vertex  $\sigma_0^0$  (call this summand  $H$ .) We then conclude that  $H \cap \ker(H_m(\Gamma_{n-1}) \rightarrow H_m(\Gamma_n)) = 0$ , i.e.,  $H_m(\Gamma_{n-1}) \rightarrow H_m(\Gamma_n)$  is injective.  $\square$

## 4 Congruence subgroups of general linear groups

For an ideal  $\mathfrak{a}$  of a ring  $R$ , we will now use Theorem 3.1 to prove that, under some conditions, the congruence subgroups  $\text{GL}_n(R, \mathfrak{a})$  exhibit homological stability, with appropriate coefficients. The theorem we will obtain was first proved by Charney [4]. The first condition we need is that  $R$  should satisfy a stable range condition. Although we will never directly use the definition of a stable range condition, we give it here for completeness.

**Definition.** Given a ring  $R$ , we call a vector  $(x_1, \dots, x_k)$  unimodular if there exist  $a_1, \dots, a_k \in R$  such that  $a_1x_1 + \dots + a_kx_k = 1$ .  $R$  is said to satisfy the stable range condition  $SR_k$  if for every unimodular vector  $(x_1, \dots, x_{k+1})$  we can find  $b_1, \dots, b_k \in R$  such that  $(x_1 + b_1x_{k+1}, x_2 + b_2x_{k+1}, \dots, x_k + b_kx_{k+1})$  is unimodular.

If  $R$  is a ring satisfying a stable range condition, then the groups  $\text{GL}_n(R)$  satisfy a number of properties; a comprehensive reference is [5], while a classical reference is [1]. Crucially for the theorems mentioned in the introduction: any commutative Noetherian ring of Krull dimension  $k - 1$  satisfies the stable range condition  $SR_k$  [5, 4.1.10]. Bass also proves the following lemma [1, V.3.2], which we will use extensively.

**Lemma 4.1.** *If  $R$  satisfies  $SR_k$ , so does  $R/\mathfrak{a}$ .*



Before stating the theorem, we introduce a few more concepts.

**Definition.** Let  $E_{ij}(r) \in \text{GL}_n(R)$  be the matrix which equals the identity except at entry  $i, j$ , where it has the value  $r$ , and write  $E_{ij}$  for  $E_{ij}(1)$ . Such a matrix is called *an elementary matrix*, and we define  $\text{EGL}_n(R)$  as the group generated by elementary matrices.

In order to apply Theorem 3.1 we will need an appropriate simplicial complex. We will first define  $\mathcal{SU}(R^n)$ , the poset of split unimodular sequences in  $R^n$ . The properties of this poset were explored in [4], and in the language of that paper, it would be written as  $\text{SU}_R(R^{n,0})$ .

**Definition.** The *poset of split unimodular sequences*  $\mathcal{SU}(R^n)$  is the poset whose elements correspond to sequences of ordered pairs

$$((x_1, y_1), \dots, (x_p, y_p))$$

where:

- $(x_i, y_i) \in R^n \times R^n$
- $x_1 \dots, x_p$  freely span a direct summand of  $R^n$
- $y_1, \dots, y_p$  freely span a direct summand of  $R^n$
- $x_i \cdot y_j = \delta_j^i$  (here  $\cdot$  denotes the standard dot product.)

The order is given by saying that  $\sigma < \tau$  if  $\sigma$  is a subsequence of  $\tau$ .

There is a left action of  $\text{GL}_n$  on  $\mathcal{SU}(R^n)$  given by

$$T \cdot ((x_1, y_1), \dots, (x_p, y_p)) = ((Tx_1, T^*y_1), \dots, (Tx_p, T^*y_p))$$

where  $T^* = (T^T)^{-1}$ . Observe that the space of vectors perpendicular to all of the  $y_i$  is a complement to the span of the  $x_i$  in  $R^n$ . Furthermore, if  $u \in R^n$  is perpendicular to all the  $y_i$ , then  $Tu$  is perpendicular to all the  $T^*y_i$ . One should thus think of the  $y_i$  as determining a distinguished complement to the span of the  $x_i$ . Unfortunately, Theorem 3.1 demands a simplicial complex, but  $\mathcal{SU}(R^n)$  is not one.

**Definition.** The *complex of unordered split unimodular sequences*  $\tilde{\mathcal{SU}}(R^n)$  is the simplicial complex whose  $p$ -cells correspond to sets of ordered pairs  $\{(x_1, y_1), \dots, (x_{p+1}, y_{p+1})\}$  where:

- $(x_i, y_i) \in R^n \times R^n$

- $x_0, \dots, x_p$  freely span a direct summand of  $R^n$
- $y_0, \dots, y_p$  freely span a direct summand of  $R^n$
- $x_i \cdot y_j = \delta_j^i$  (here  $\cdot$  denotes the standard dot product.)

Incidence is determined by saying that  $\sigma = \{(x_1, y_1), \dots\}$  is a face of  $\tau = \{(u_1, v_1), \dots\}$  if  $\{(x_1, y_1), \dots\}$  is a subset of  $\{(u_1, v_1), \dots\}$ .

There is a left action of  $\mathrm{GL}_n$  on  $\tilde{\mathcal{SU}}(R^n)$  given by

$$T \cdot \{(x_1, y_1), \dots, (x_{p+1}, y_{p+1})\} = \{(Tx_1, T^*y_1), \dots, (Tx_p, T^*y_p)\}$$

where  $T^* = (T^T)^{-1}$ . There is a natural map (of posets)  $g : \mathcal{SU}(R^n) \rightarrow \tilde{\mathcal{SU}}(R^n)$  given by forgetting the order of the elements in a sequence. As in [13, Lemma 4.1], we observe that this map has a right inverse  $f : \tilde{\mathcal{SU}}(R^n) \rightarrow \mathcal{SU}(R^n)$  given as follows: take any total order  $\prec$  on the elements of  $R^n \times R^n$ . Then  $f$  maps a simplex  $\sigma^p = \{(x_1, y_1), \dots, (x_{p+1}, y_{p+1})\} \in \tilde{\mathcal{SU}}$  to  $((x_{i_1}, y_{i_1}), \dots, (x_{i_{p+1}}, y_{i_{p+1}}))$  where

$$(x_{i_1}, y_{i_1}) \prec (x_{i_2}, y_{i_2}) \prec \dots \prec (x_{i_{p+1}}, y_{i_{p+1}}).$$

The map  $f$  is easily seen to be order preserving. Since  $g \circ f$  is the identity,  $g$  induces a surjection on homology. We have established the following lemma.

**Lemma 4.2.** *If  $\tilde{H}_q(\mathcal{SU}(R^n); M) = 0$  for  $q = 0, \dots, m$ , then*

$$\tilde{H}_q(\tilde{\mathcal{SU}}(R^n); M) = 0$$

for  $q = 0, \dots, m$ .

We are now ready to state and prove homological stability for congruence subgroups. In our notation, we generally consider vectors to be column matrices, for instance, if  $e_1, \dots, e_n$  is the standard basis for  $R^n$ , we write the  $n \times n$  identity matrix as

$$\begin{bmatrix} e_1 & \dots & e_n \end{bmatrix}.$$

**Theorem 4.3.** *Suppose that  $R$  is a ring satisfying the stable range condition  $SR_k$ , and  $\mathfrak{a}$  an ideal of  $R$ . Suppose further that the inclusion of additive groups  $\mathfrak{a} \subset R$  induces an isomorphism:  $H_*(\mathfrak{a}; M) \xrightarrow{\cong} H_*(R; M)$ . Then for  $n \geq 2m + k + 4$ ,*

$$H_m(\mathrm{GL}_{n-1}(R, \mathfrak{a})) \rightarrow H_m(\mathrm{GL}_n(R, \mathfrak{a}))$$

is a surjection, and for  $n \geq 2m + k + 5$ , this map is an isomorphism.

*Proof.* From here on, as in the previous section,  $H_*(X)$  will mean  $H_*(X; M)$ . As promised, we will apply Theorem 3.1. We will use:

- $G_n = \text{EGL}_n(R) \text{GL}_n(R, \mathfrak{a})$
- $\Gamma_n = \text{GL}_n(R, \mathfrak{a})$
- $X_n = \tilde{\mathcal{S}}\mathcal{U}(R^n)$ , and  $d = k + 3$

where  $G_{n-1} \rightarrow G_n$  and  $\Gamma_{n-1} \rightarrow \Gamma_n$  are given by the lower right inclusion

$$A \mapsto \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}.$$

We must verify all of the assumptions of Theorem 3.1.

**Assumption 1.**  $G_n$  acts transitively on  $p$ -cells of  $X_n$

[4, 3.2] verifies that  $\text{EGL}_n(R)$  acts transitively on  $\mathcal{S}\mathcal{U}(R^n)$ , so it certainly acts transitively on  $\tilde{\mathcal{S}}\mathcal{U}(R^n)$ .

**Assumption 2.** For all  $p = 0, \dots, n$ , there is a standard  $p$ -cell  $\sigma_0^p$  in  $X_n$  such that, for  $n > p + d$ ,  $\text{Stab}_{\Gamma_n} \sigma_0^p = \Gamma_{n-p-1}$ , and furthermore,  $\text{Stab}_{\Gamma_n} \sigma_0^p$  fixes the vertices of  $\sigma_0^p$ .

The standard  $p$ -simplex of  $\tilde{\mathcal{S}}\mathcal{U}(R^n)$  is given by:

$$\sigma_0^p = \{(e_1, e_1), (e_2, e_2), \dots, (e_{p+1}, e_{p+1})\}$$

where  $e_1 = (1, 0, \dots, 0)$ ,  $e_2 = (0, 1, 0, \dots, 0)$ ,  $\dots$ . If a matrix fixes this simplex, then its first  $p + 1$  columns must be (in no particular order)  $e_1, e_2, \dots, e_{p+1}$ . The remaining columns must be perpendicular to  $e_1, \dots, e_{p+1}$ . When we consider also that the  $j$ -th column of an element of  $\Gamma_n$  must be congruent to  $e_j \bmod \mathfrak{a}$ , we see that  $\text{Stab}_{\Gamma_n}(\sigma_0^p) = \Gamma_{n-p-1}$ , and this stabilizer fixes all the vertices of  $\sigma_0^p$ .

**Assumption 3.**  $G_n$  acts trivially on the image of the map  $H_p(\Gamma_{n-1}) \rightarrow H_p(\Gamma_n)$  induced by inclusion  $\Gamma_{n-1} \rightarrow \Gamma_n$  for  $n > d$  and  $p \geq 0$ .

This is proved in [4], but we reproduce the proof here for completeness. To verify the assumption, we must prove that  $\text{EGL}_n(R) \text{GL}_n(R, \mathfrak{a})$  acts trivially on  $H = \text{image}(H_p(\text{GL}_{n-1}(R, \mathfrak{a})) \rightarrow H_p(\text{GL}_n(R, \mathfrak{a})))$  for  $n > k + 3$ . First, observe that  $\text{GL}_n(R, \mathfrak{a})$  acts trivially on its own homology, so we just need to show that the elementary matrices act trivially on  $H$ .

Let  $S = \{E_{1j}(r)|r \in R\} \cup \{E_{i1}(r)|r \in R\}$ . We will show that  $S$  generates  $\text{EGL}_n(R)$ , then we will show that  $S$  acts trivially on  $H$ . Observe that if  $i, j, k$  distinct we have the commutator identity  $[E_{ij}(r), E_{jk}(s)] = E_{ik}(rs)$ , so  $S$  generates  $\text{EGL}_n(R)$  (observe that we have used the fact that  $n \geq 3$ ).

Now we show that  $E_{1j}(r)$  acts trivially on  $H$  (the  $E_{i1}$  case is similar). Let

$$G = \left\{ \begin{bmatrix} 1 & v^T \\ 0 & A \end{bmatrix} \in \text{GL}_n(R) | v \in \mathfrak{a}^{n-1}, G \in \text{GL}_{n-1}(R, \mathfrak{a}) \right\}$$

and

$$\hat{G} = \left\{ \begin{bmatrix} 1 & v^T \\ 0 & A \end{bmatrix} \in \text{GL}_n(R) | v \in R^{n-1}, G \in \text{GL}_{n-1}(R, \mathfrak{a}) \right\}.$$

We have that  $\text{GL}_{n-1}(R, \mathfrak{a}) \subset G \subset \text{GL}_n(R, \mathfrak{a})$ , so it suffices to show that  $E_{1j}(r) \in \hat{G}$  acts trivially on  $H_*(G)$  since  $H_p(\text{GL}_{n-1}(R, \mathfrak{a})) \rightarrow H_p(\text{GL}_n(R, \mathfrak{a}))$  factors through  $H_p(G) \rightarrow H_p(\text{GL}_n(R, \mathfrak{a}))$ . But we know that  $E_{1j}(r)$  acts trivially on  $H_p(\hat{G})$  since  $E_{1j}(r) \in \hat{G}$ . So we need only show that the  $\text{GL}_n(R)$ -equivariant map  $H_p(G) \rightarrow H_p(\hat{G})$  is an isomorphism.

Consider the map  $G \rightarrow \text{GL}_{n-1}(R, \mathfrak{a})$  given by:

$$\begin{bmatrix} 1 & v^T \\ 0 & A \end{bmatrix} \mapsto A$$

This map has kernel isomorphic to  $\mathfrak{a}^{n-1}$ , and similarly  $\hat{G} \rightarrow \text{GL}_{n-1}(R, \mathfrak{a})$  has kernel isomorphic to  $R^{n-1}$ . Thus we have a commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathfrak{a}^{n-1} & \longrightarrow & G & \longrightarrow & \text{GL}_{n-1}(R, \mathfrak{a}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & R^{n-1} & \longrightarrow & \hat{G} & \longrightarrow & \text{GL}_{n-1}(R, \mathfrak{a}) \longrightarrow 1 \end{array}$$

This spectral sequence is natural. Since  $\mathfrak{a}^{n-1} \rightarrow R^{n-1}$  induces an isomorphism on homology, we have that  $H_*(G) \rightarrow H_*(\hat{G})$  is an isomorphism as desired, by the following lemma.

**Lemma 4.4.** *Suppose we have a diagram of groups, with exact rows,*

$$\begin{array}{ccccccc} 1 & \longrightarrow & N_1 & \longrightarrow & G_1 & \longrightarrow & Q \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & N_2 & \longrightarrow & G_2 & \longrightarrow & Q \longrightarrow 1 \end{array}$$

If  $N_1 \rightarrow N_2$  induces an isomorphism on homology, and the action of  $Q$  on  $H_p(N_1)$  and  $H_p(N_2)$  is trivial, then

$$H_p(G_1) \rightarrow H_p(G_2)$$

is an isomorphism.

*Proof.* Given a short exact sequence of groups  $1 \rightarrow N \rightarrow G \rightarrow Q$ , there is a spectral sequence called the Hochschild-Serre spectral sequence [3, VII.6] with  $E_{pq}^2 = H_p(Q; H_q(N))$  and  $E_{pq}^\infty \Rightarrow H_{p+q}(G)$ . This spectral sequence is natural, in the sense that our diagram induces a map from the spectral sequence associated to its top row to the spectral sequence associated to its bottom row. On  $E_{pq}^2$  this takes the form of the natural map

$$H_p(Q; H_q(N_1)) \rightarrow H_p(Q; H_q(N_2)).$$

So if this map is an isomorphism, we see that the map of abutments is an isomorphism, i.e.,  $H_p(G_1) \cong H_p(G_2)$ .  $\square$

**Assumption 4.**  $X_n$  and  $X_n/\Gamma_n$  are  $\frac{n-d}{2}$  acyclic.

It is proved in [4, 3.5] that  $\mathcal{SU}(R^n)$  is  $\frac{n-d}{2}$  acyclic. Lemma 4.2 thus implies that  $X_n$  is  $\frac{n-d}{2}$  acyclic, so we only need to worry about  $X_n/\Gamma_n$ . Given  $\sigma_p = \{(x_1, y_1), \dots, (x_{p+1}, y_{p+1})\}$  a  $p$ -simplex of  $\tilde{\mathcal{SU}}(R^n)$ , it is evident that for  $T \in \Gamma_n$ , we have  $T \cdot (x_i, y_i) \equiv (x_i, y_i) \pmod{\mathfrak{a}}$ . Thus, one might guess that  $X_n/\Gamma_n$  is  $\tilde{\mathcal{SU}}((R/\mathfrak{a})^n)$ . This is mostly correct. We will show that the  $n-d$  skeleton  $(X_n/\Gamma_n)^{(n-d)} = \tilde{\mathcal{SU}}((R/\mathfrak{a})^n)^{(n-d)}$ , which is  $\frac{n-d}{2}$  connected by Lemma 4.2. Let  $\phi$  denote the natural map

$$X_n^{n-d}/\Gamma_n \rightarrow \tilde{\mathcal{SU}}((R/\mathfrak{a})^n)^{n-d}$$

which takes the orbit of a simplex  $\{(x_1, y_1), \dots\}$  to the simplex  $\{(\bar{x}_1, \bar{y}_1), \dots\}$  (where  $x \mapsto \bar{x}$  denotes reduction mod  $\mathfrak{a}$ ). Since  $\phi$  clearly respects incidence, we just need to show that  $\phi$  is a bijection.

**Surjectivity of  $\phi$ .** In establishing that  $\phi$  is surjective, our starting point is the observation that  $\text{EGL}_n(R) \rightarrow \text{EGL}_n(R/\mathfrak{a})$  is surjective. By Lemma 4.1  $R/\mathfrak{a}$  satisfies  $SR_k$ , so we know that  $\text{EGL}_n(R/\mathfrak{a})$  acts transitively on  $p$ -simplices of  $\tilde{\mathcal{SU}}((R/\mathfrak{a})^n)^{(n-d)}$  by [4, 3.2]. Thus, given  $\tau^p$  a  $p$ -simplex of  $\tilde{\mathcal{SU}}((R/\mathfrak{a})^n)^{(n-d)}$ , we can find  $\bar{T} \in \text{EGL}_n(R/\mathfrak{a})$  such that  $\bar{T}\phi(\sigma_0^p) = \tau^p$ . But then there exists  $T \in \text{EGL}_n(R)$  which reduces to  $\bar{T} \pmod{\mathfrak{a}}$ , and thus  $\phi(T\sigma_0^p) = \tau^p$ .

**Injectivity of  $\phi$ .** It suffices to show that for any  $p < n - d$  and any length  $p+1$  split unimodular sequence  $\tau^p = ((x_1, y_1), \dots, (x_{p+1}, y_{p+1})) \equiv \sigma_0^p \pmod{\mathfrak{a}}$ , there exists  $T \in \mathrm{GL}_n(R, \mathfrak{a})$  such that  $T\sigma_0^p = \tau^p$ .

Charney [4, 3.2] proves that there is some  $S \in \mathrm{EGL}_n(R)$  such that  $S\sigma_0^p = \tau^p$ . Observe that  $S$  has columns

$$\begin{bmatrix} x_1 & \dots & x_{p+1} & * & \dots & * \end{bmatrix}$$

and  $S^{-1}$  has rows

$$\begin{bmatrix} y_1^T \\ \vdots \\ y_{p+1}^T \\ * \\ \vdots \\ * \end{bmatrix}.$$

Let  $\bar{S}$  be the image of  $S$  under  $\mathrm{GL}_n(R) \rightarrow \mathrm{GL}_n(R/\mathfrak{a})$ . Then because  $\tau^p \equiv \sigma_0^p \pmod{\mathfrak{a}}$ , we know  $\bar{S}$  has the form:

$$\begin{bmatrix} \mathrm{Id}_{p+1} & 0 \\ 0 & A \end{bmatrix}$$

since the first  $p+1$  columns of  $\bar{S}$  must be  $e_1, \dots, e_{p+1}$  and the remaining columns must be perpendicular to  $e_1, \dots, e_{p+1}$ .

By [5, 4.2.13], the map:

$$\mathrm{GL}_{n-p-1}(R/\mathfrak{a})/\mathrm{EGL}_{n-p-1}(R/\mathfrak{a}) \rightarrow \mathrm{GL}_n(R/\mathfrak{a})/\mathrm{EGL}_n(R/\mathfrak{a})$$

induced by lower right inclusion is an injection since  $n - p - 1$  exceeds the stable range of  $R/\mathfrak{a}$ . So, since  $\bar{S} \in \mathrm{EGL}_n(R/\mathfrak{a})$ , we conclude that  $A \in \mathrm{EGL}_{n-p-1}(R/\mathfrak{a})$ . But  $\mathrm{EGL}_n(R) \rightarrow \mathrm{EGL}_n(R/\mathfrak{a})$  is surjective, so we can lift  $A$  to some matrix  $\tilde{A} \in \mathrm{EGL}_{n-p-1}(R)$ . Let

$$\tilde{S} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A} \end{bmatrix}$$

and  $T = S\tilde{S}^{-1}$ . Then clearly  $T \in \mathrm{GL}_n(R, \mathfrak{a})$  (since  $S \cong \tilde{S} \pmod{\mathfrak{a}}$ ), and  $T\sigma_0^p = S\tilde{S}^{-1}\sigma_0^p = S\sigma_0^p = \tau^p$  as desired, verifying the final assumption of Theorem 3.1.  $\square$

## 5 Congruence subgroups of symplectic groups

Now we prove a very similar theorem for congruence subgroups of symplectic groups. However, we will need  $R$  to satisfy a stronger sort of stable range condition. We begin by defining this condition, as well as a number of other symplectic analogues of previously considered concepts.

**Definition.** Let  $R$  be a commutative ring. Suppose  $x = (x_1, \dots, x_{2n})$  and  $y = (y_1, \dots, y_{2n})$  are vectors in  $R^{2n}$ . We say that the symplectic product  $\langle x, y \rangle$  is  $x_1 y_{n+1} + x_2 y_{n+2} + \dots + x_n y_{2n} - x_{n+1} y_1 - x_{n+2} y_2 - \dots - x_{2n} y_n$ .

The symplectic group  $\mathrm{Sp}_{2n}(R)$  is the subgroup of  $\mathrm{GL}_{2n}(R)$  which preserves the symplectic form, i.e.,  $\mathrm{Sp}_{2n}(R)$  is the set of  $T \in \mathrm{GL}_{2n}(R)$  such that for all  $x, y \in R^{2n}$ , we have  $\langle Tx, Ty \rangle = \langle x, y \rangle$ . Now we will define elementary symplectic matrices. Let  $e_{ij}(r)$  denote the  $n \times n$  matrix with all entries 0 except that entry  $i, j$  is equal to  $r$ . Let  $E_{ij}(r) = \mathrm{Id}_n + e_{ij}(r)$ .

**Definition.** The following types of matrices are called elementary symplectic matrices (in all cases  $i \neq j$ ):

$$\begin{aligned} A_{ij}(r) &= \begin{bmatrix} E_{ij}(r) & 0 \\ 0 & E_{ji}(-r) \end{bmatrix} \\ B_{ij}(r) &= \begin{bmatrix} \mathrm{Id}_n & e_{ij}(r) + e_{ji}(r) \\ 0 & \mathrm{Id}_n \end{bmatrix} \\ B_{ii}(r) &= \begin{bmatrix} \mathrm{Id}_n & e_{ii}(r) \\ 0 & \mathrm{Id}_n \end{bmatrix} \\ C_{ij}(r) &= \begin{bmatrix} \mathrm{Id}_n & 0 \\ e_{ij}(r) + e_{ji}(r) & \mathrm{Id}_n \end{bmatrix} \\ C_{ii}(r) &= \begin{bmatrix} \mathrm{Id}_n & 0 \\ e_{ii}(r) & \mathrm{Id}_n \end{bmatrix} \end{aligned}$$

The group generated by these types of matrices is called the *elementary symplectic group*  $\mathrm{ESp}_{2n}(R)$ .

**Definition.** Suppose  $\mathfrak{a}$  is an ideal of  $R$ . The symplectic congruence subgroup  $\mathrm{Sp}_{2n}(R, \mathfrak{a})$  is the kernel of the natural map  $\mathrm{Sp}_{2n}(R) \rightarrow \mathrm{Sp}_{2n}(R/\mathfrak{a})$ .

**Definition.** A partial symplectic basis of  $R^{2n}$  is a sequence of pairs of vectors  $(a_1, b_1), (a_2, b_2), \dots, (a_p, b_p)$  where  $a_i, b_i \in R^{2n}$ , and we have that, for all  $i, j$  the relations  $\langle a_i, a_j \rangle = 0$  and  $\langle b_i, b_j \rangle = 0$  hold, and  $\langle a_i, b_j \rangle = \delta_j^i$ .

The poset of partial symplectic bases  $\mathcal{HU}(R^{2n})$  is the poset whose elements are partial symplectic bases. Order is given by  $\sigma < \tau$  whenever  $\sigma$  is a subsequence of  $\tau$ . The notation  $\mathcal{HU}$  arises from the fact that this is a special case of the complex of hyperbolic bases as considered in [9, §7].

**Definition.** An unordered partial symplectic basis of  $R^{2n}$  is a set of pairs of vectors  $\{(a_1, b_1), (a_2, b_2), \dots, (a_p, b_p)\}$  where  $a_i, b_i \in R^{2n}$ , and we have that, for all  $i, j$  the relations  $\langle a_i, a_j \rangle = 0$  and  $\langle b_i, b_j \rangle = 0$  hold, and  $\langle a_i, b_j \rangle = \delta_j^i$ . The complex of unordered partial symplectic bases  $\tilde{\mathcal{HU}}(R^{2n})$  is the simplicial complex whose  $p$ -simplices correspond to partial symplectic bases of length  $p$ . Incidence is given by saying that a simplex  $\sigma$  is the face of a simplex  $\tau$  whenever  $\sigma$  is a subsequence of  $\tau$ .

The following lemma is proved exactly as Lemma 4.2.

**Lemma 5.1.** *If  $\tilde{H}_q(\mathcal{HU}(R^n); M) = 0$  for  $q = 0, \dots, m$ , then*

$$\tilde{H}_q(\tilde{\mathcal{HU}}(R^n); M) = 0$$

*for  $q = 0, \dots, m$ .*

**Definition.** A commutative ring  $R$  is said to satisfy the symplectic stable range condition  $SpSR_k$  if it satisfies  $SR_k$  and  $\mathrm{ESp}_{2(k+1)}(R)$  acts transitively on the unimodular vectors of  $R^{2(k+1)}$ . This is a special case of the unitary stable range condition of [9, 6.3].

Importantly, any commutative noetherian ring of Krull dimension  $k - 1$  satisfies  $SpSR_k$  [9, 6.5].

**Lemma 5.2.** *If  $R$  satisfies  $SpSR_k$ , so does  $R/\mathfrak{a}$ .*

*Proof.* The proof of [1, V.3.2] shows that a unimodular vector of  $(R/\mathfrak{a})^{2(n+1)}$  lifts to a unimodular vector of  $R^{2(n+1)}$ . The lemma thus follows immediately when we notice that  $\mathrm{ESp}_{2(n+1)}(R) \rightarrow \mathrm{ESp}_{2(n+1)}(R/\mathfrak{a})$  is surjective.  $\square$

**Theorem 5.3.** *Suppose that  $R$  is a ring satisfying the symplectic stable range condition  $SpSR_k$ , and  $\mathfrak{a}$  an ideal of  $R$ . Suppose further that the inclusion of groups  $\mathfrak{a} \subset R$  induces an isomorphism  $H_*(\mathfrak{a}; M) \xrightarrow{\cong} H_*(R; M)$ . Then for  $n \geq 2m + k + 4$ , the map*

$$H_m(\mathrm{Sp}_{2(n-1)}(R, \mathfrak{a})) \rightarrow H_m(\mathrm{Sp}_{2n}(R, \mathfrak{a}))$$

*is a surjection, and for  $n \geq 2m + k + 5$ , this map is an isomorphism. From here on, as always,  $H_*(X)$  will mean  $H_*(X; M)$ .*



*Proof.* We will apply Theorem 3.1 with

- $G_n = \mathrm{ESp}_{2n}(R) \mathrm{Sp}_{2n}(R, \mathfrak{a})$
- $\Gamma_n = \mathrm{Sp}_{2n}(R, \mathfrak{a})$
- $X_n = \mathcal{HU}(R^{2n})$ ,  $d = k + 3$ .

We must verify that all the assumptions are valid.

**Assumption 1.**  $G_n$  acts transitively on  $p$ -simplices of  $X_n$

[9, Lemma 7.1] proves that  $\mathrm{ESp}_{2n}(R)$  acts transitively on  $p$ -simplices of  $\mathcal{HU}(R^{2n})$  for  $p < n - d$ , which suffices to prove the assumption.

**Assumption 2.** For all  $p = 0, \dots, n$ , there is a standard  $p$ -simplex  $\sigma_0^p$  in  $X_n$  such that, for  $n > p + d$ ,  $\mathrm{Stab}_{\Gamma_n} \sigma_0^p = \Gamma_{n-p-1}$ , and furthermore,  $\mathrm{Stab}_{\Gamma_n} \sigma_0^p$  fixes the vertices of  $\sigma_0^p$ .

The standard  $p$ -cell  $\sigma_0^p$  is the standard hyperbolic partial basis:

$$\sigma_0^p = \{(e_1, e_{n+1}), (e_2, e_{n+2}), \dots, (e_{p+1}, e_{n+p+1})\}$$

Since a matrix in  $\mathrm{Sp}_{2n}(R, \mathfrak{a})$  which permutes  $e_1, \dots, e_{p+1}$  must fix these vectors, we have that the stabilizer of  $\sigma_0^p$  in  $\mathrm{Sp}_{2n}(R, \mathfrak{a})$  consists of all matrices of the form:

$$\begin{bmatrix} \mathrm{Id}_{p+1} & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & \mathrm{Id}_{p+1} & 0 \\ 0 & * & 0 & * \end{bmatrix} \in \mathrm{Sp}_{2n}(R, \mathfrak{a})$$

which is clearly just (the lower-right-included image of)  $\mathrm{Sp}_{2(n-p-1)}(R)$ . Thus the assumption is verified.

**Assumption 3.**  $G_n$  acts trivially on the image of the map  $H_p(\Gamma_{n-1}) \rightarrow H_p(\Gamma_n)$  induced by inclusion  $\Gamma_{n-1} \rightarrow \Gamma_n$  for  $n > d$  and  $p \geq 0$ .

Our approach here is similar to the general linear case. It suffices to show that elementary symplectic matrices act trivially on the image  $H$  of  $H_p(\mathrm{Sp}_{2(n-1)}(R)) \rightarrow H_p(\mathrm{Sp}_{2n}(R))$  for  $n > d$ . We first prove a small lemma:

**Lemma 5.4.** Matrices of the types  $A_{1j}(r)$ ,  $A_{i1}(r)$ ,  $B_{1j}(r)$ , and  $C_{i1}(r)$  generate  $\mathrm{ESp}_{2n}(R)$ .

*Proof.* For alternating permutations  $\pi \in A_n$ , let  $T_\pi \in \text{GL}_n(R)$  be such that  $T_\pi e_i = e_{\pi(i)}$ . We have already observed that  $T_\pi$  can be generated by matrices of the forms  $E_{i1}$  and  $E_{1j}$ , so it follows that the matrix:

$$S_\pi = \begin{bmatrix} T_\pi & 0 \\ 0 & T_\pi \end{bmatrix}$$

can be generated by matrices of the forms  $A_{1j}(1)$  and  $A_{i1}(1)$ . But then by conjugating matrices of types  $A_{1j}(r)$ ,  $A_{i1}(r)$ ,  $B_{1j}(r)$  and  $C_{1j}(r)$  by the  $S_\pi$ , we can generate every elementary symplectic matrix. For instance  $B_{23}(r) = S_{(123)}B_{12}(r)S_{(123)}^{-1}$ .  $\square$

Now we will show that matrices of types  $A_{1j}(r)$  and  $B_{1j}(r)$  act trivially on  $H$ . The case for  $A_{i1}(r)$  and  $C_{i1}(r)$  is similar. Let:

$$G = \left\{ \begin{bmatrix} 1 & * & * & * \\ 0 & A & * & B \\ 0 & 0 & 1 & 0 \\ 0 & C & * & D \end{bmatrix} \in \text{Sp}_{2n}(R, \mathfrak{a}) \right\}$$

$$\hat{G} = \left\{ \begin{bmatrix} 1 & * & * & * \\ 0 & A & * & B \\ 0 & 0 & 1 & 0 \\ 0 & C & * & D \end{bmatrix} \in \text{Sp}_{2n}(R) \mid \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_{2(n-1)}(R, \mathfrak{a}) \right\}.$$

We have that  $A_{1j}(r), B_{1j}(r) \in \hat{G}$ , so these matrices act trivially on  $H_p(\hat{G})$ . On the other hand  $\text{Sp}_{2(n-1)}(R, \mathfrak{a}) \subset G \subset \text{Sp}_{2n}(R, \mathfrak{a})$ , so it suffices to show that  $A_{1j}(r), B_{1j}(r)$  act trivially on  $H_p(G)$ . This will follow once we show that  $H_p(G) \rightarrow H_p(\hat{G})$  is an isomorphism. Let  $K$  be the kernel of the natural map  $G \rightarrow \text{Sp}_{2(n-1)}$  given by forgetting rows  $1, n+1$  and columns  $1, n+1$ . Similarly, let  $\hat{K} = \ker(\hat{G} \rightarrow \text{Sp}_{2(n-1)})$ . Then we have a commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & \text{Sp}_{2(n-1)} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \hat{K} & \longrightarrow & \hat{G} & \longrightarrow & \text{Sp}_{2(n-1)} \longrightarrow 1 \end{array}$$

So it suffices to show that  $K \rightarrow \hat{K}$  induces an isomorphism on homology,

which follows from applying lemma 4.4 to the commutative diagram:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathfrak{a} & \longrightarrow & K & \longrightarrow & \mathfrak{a}^{2(n-1)} \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \parallel \\
1 & \longrightarrow & R & \longrightarrow & \hat{K} & \longrightarrow & R^{2(n-1)} \longrightarrow 1
\end{array}$$

Thus, the assumption is verified.

**Assumption 4.**  $X_n$  and  $X_n/\Gamma_n$  are  $\frac{n-d}{2}$  connected.

[9, Lemma 7.4] shows that  $\mathcal{HU}(R^{2n})$  is  $\frac{n-k-3}{2}$  connected for  $R$  satisfying  $SpSR_k$ . So it suffices to prove that the  $n-d$ -skeleton  $(\mathcal{HU}(R^{2n})/\Gamma_n)^{(n-d)} = \mathcal{HU}((R/\mathfrak{a})^{2n})^{(n-d)}$ . We want to show that  $\mathrm{Sp}_{2n}(R, \mathfrak{a})$  orbits of unordered partial hyperbolic bases of  $R^{2n}$  correspond to  $\mathfrak{a}$ -congruence classes of unordered partial hyperbolic bases of  $R^{2n}$ . That is, given  $\sigma^p, \tau^p$  partial hyperbolic bases of length  $p+1$ , we want that:

$$\sigma^p \equiv \tau^p \bmod \mathfrak{a} \iff \sigma^p \in \mathrm{Sp}_{2n}(R, \mathfrak{a})\tau^p$$

The  $\Leftarrow$  implication is trivial, so we now prove the  $\Rightarrow$  implication. It suffices to prove this for ordered, rather than unordered, partial hyperbolic bases. Let  $\sigma_0^p$  denote the standard hyperbolic partial basis

$$((e_1, e_{n+1}), (e_2, e_{n+2}), \dots, (e_{p+1}, e_{n+p+1})).$$

We must show that for any length  $p+1$  partial hyperbolic basis  $\tau^p = ((x_1, y_1), \dots, (x_{p+1}, y_{p+1})) \equiv \sigma_0^p \bmod \mathfrak{a}$ , there exists  $T \in \mathrm{Sp}_{2n}(R, \mathfrak{a})$  such that  $T\sigma_0^p = \tau^p$ . As we observed before, [9] proves that there is some  $S \in \mathrm{ESp}_{2n}(R)$  such that  $S\sigma_0^p = \tau^p$  and thus  $S$  has the form

$$\begin{bmatrix} x_1 & \dots & x_{p+1} & * & y_1 & \dots & y_{p+1} & * \end{bmatrix}$$

Let  $\bar{S}$  be the image of  $S$  under  $\mathrm{Sp}_{2n}(R) \rightarrow \mathrm{Sp}_{2n}(R/\mathfrak{a})$ . Then  $\bar{S}$  has the form:

$$\begin{bmatrix} \mathrm{Id}_{p+1} & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & \mathrm{Id}_{p+1} & 0 \\ 0 & C & 0 & D \end{bmatrix}$$

By [5, 9.1.11], the map:

$$\mathrm{Sp}_{2(n-p-1)}(R/\mathfrak{a})/\mathrm{ESp}_{2(n-p-1)}(R/\mathfrak{a}) \rightarrow \mathrm{Sp}_{2n}(R/\mathfrak{a})/\mathrm{ESp}_{2n}(R/\mathfrak{a})$$

induced by lower right inclusion is an injection since  $n - p - 1$  exceeds the stable range of  $R/\mathfrak{a}$ . So, since  $\tilde{S} \in \mathrm{ESp}_{2n}(R/\mathfrak{a})$ , we conclude that  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{ESp}_{2(n-p-1)}(R/\mathfrak{a})$ . But  $\mathrm{ESp}_{2n}(R) \rightarrow \mathrm{ESp}_{2n}(R/\mathfrak{a})$  is surjective, so we can lift  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  to some matrix  $\tilde{S} \in \mathrm{ESp}_{2(n-p-1)}(R) \subset \mathrm{ESp}_{2n}(R)$ . Let  $T = S\tilde{S}^{-1}$ . Then clearly  $T \in \mathrm{Sp}_{2n}(R, \mathfrak{a})$ , and  $T\sigma_0^p = S\tilde{S}^{-1}\sigma_0^p = S\sigma_0^p = \tau^p$  as desired. This completes the verification of the final assumption, and thus, the proof.  $\square$

## 6 Borel Stability

Using theorem 4.3 (and its proof) we will obtain a Borel stability result.

**Theorem 6.1.** *Suppose that  $R$  is a ring satisfying the stable range condition  $SR_k$ , and  $\mathfrak{a}$  an ideal of  $R$ , with  $R/\mathfrak{a}$  finite. Then for  $m \geq 1$  and  $n \geq 2m + 2k + 4$ , the map:*

$$H_m(\mathrm{GL}_n(R, \mathfrak{a}); \mathbb{Q}) \rightarrow H_m(\mathrm{GL}_n(R); \mathbb{Q})$$

*is an isomorphism.*

*Proof.* First, observe that because  $\mathrm{GL}_n(R/\mathfrak{a})$  is finite, we have that:

$$[\mathrm{GL}_n(R) : \mathrm{GL}_n(R, \mathfrak{a})] < \infty$$

Now we apply [3, III.10.4] to see that the rational homology of  $\mathrm{GL}_n(R)$  is isomorphic to the  $\mathrm{GL}_n(R)$ -coinvariants of the rational homology of  $\mathrm{GL}_n(R, \mathfrak{a})$ . In other words,

$$H_*(\mathrm{GL}_n(R); \mathbb{Q}) = H_*(\mathrm{GL}_n(R, \mathfrak{a}); \mathbb{Q})_{\mathrm{GL}_n(R)}$$

So, if the action of  $\mathrm{GL}_n(R)$  on  $H_m(\mathrm{GL}_n(R, \mathfrak{a}); \mathbb{Q})$  is trivial, the desired result will follow. Observe that [5, 4.2.13] implies that

$$\mathrm{GL}_n(R) = \langle \mathrm{EGL}_n(R), \mathrm{GL}_k(R) \rangle,$$

so we just need to show that  $\mathrm{EGL}_n(R)$  and  $\mathrm{GL}_k(R)$  act trivially.

**$\mathrm{EGL}_n(R)$  acts trivially.** Because  $R/\mathfrak{a}$  is finite, and  $R$ , as an additive group, is abelian, [3, III.10.4] implies that  $H_*(\mathfrak{a}; \mathbb{Q}) \rightarrow H_*(R; \mathbb{Q})$  is an isomorphism, so we can apply Theorem 4.3 to see that  $H_m(\mathrm{GL}_{n-1}(R, \mathfrak{a}); \mathbb{Q}) \rightarrow H_m(\mathrm{GL}_n(R, \mathfrak{a}); \mathbb{Q})$  is a surjection. On the other hand, our verification of assumption 3 in our proof of Theorem 3.1 already showed that  $\mathrm{EGL}_n(R)$  acts trivially on the image of  $H_m(\mathrm{GL}_{n-1}(R, \mathfrak{a}); \mathbb{Q}) \rightarrow H_m(\mathrm{GL}_n(R, \mathfrak{a}); \mathbb{Q})$  for  $n \geq k+3$ , so we conclude that  $\mathrm{EGL}_n(R)$  acts trivially on  $H_m(\mathrm{GL}_n(R, \mathfrak{a}); \mathbb{Q})$ .

**$\mathrm{GL}_k(R)$  acts trivially.** Let  $\pi \in S_n$  be a permutation which carries  $\{n-k+1, \dots, n\}$  to  $\{1, \dots, k\}$  and vice versa. Let  $T = \mathrm{sgn}(\pi)T_\pi \in \mathrm{EGL}_n(R)$ . Then  $T \mathrm{GL}_k(R) T^{-1}$  commutes with  $\mathrm{GL}_{n-k}(R, \mathfrak{a})$  because:

$$T \mathrm{GL}_k(R) T^{-1} = \left\{ \begin{bmatrix} A & 0 \\ 0 & \mathrm{Id}_{n-k} \end{bmatrix} \mid A \in \mathrm{GL}_k(R) \right\}$$

and

$$\mathrm{GL}_{n-k}(R, \mathfrak{a}) = \left\{ \begin{bmatrix} \mathrm{Id}_k & 0 \\ 0 & A \end{bmatrix} \mid A \in \mathrm{GL}_{n-k}(R, \mathfrak{a}) \right\}$$

Thus,  $T \mathrm{GL}_k(R) T^{-1}$  acts trivially on  $H_m(\mathrm{GL}_{n-k}(R, \mathfrak{a}); \mathbb{Q})$ , and thus it acts trivially on  $H_m(\mathrm{GL}_n(R, \mathfrak{a}); \mathbb{Q})$  because:

$$H_m(\mathrm{GL}_{n-k}(R, \mathfrak{a}); \mathbb{Q}) \rightarrow H_m(\mathrm{GL}_n(R, \mathfrak{a}); \mathbb{Q})$$

is surjective (by Theorem 4.3 since  $n-k \geq 2m+2k+4-k = 2m+k+4$ .) Thus, since  $T \in \mathrm{EGL}_n(R)$  acts trivially on  $H_m(\mathrm{GL}_n(R, \mathfrak{a}))$ , so does  $\mathrm{GL}_k(R)$ , and the theorem is proved.  $\square$

Finally, we will state the symplectic version of Theorem 6.1 (the proof is analogous.)

**Theorem 6.2.** *Suppose that  $R$  is a commutative ring satisfying the symplectic stable range condition  $\mathrm{SpSR}_k$ , and  $\mathfrak{a}$  an ideal of  $R$ , with  $R/\mathfrak{a}$  finite. Then for  $m \geq 1$  and  $n \geq 2m+2k+4$ , the map:*

$$H_m(\mathrm{Sp}_{2n}(R, \mathfrak{a}); \mathbb{Q}) \rightarrow H_m(\mathrm{Sp}_{2n}(R); \mathbb{Q})$$

*is an isomorphism.*

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